A FAMILY OF COMPACTIFICATIONS ACCOUNTING FOR ALL ARGUMENTS OF INFINITY

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ABSTRACT. We study a family of compactifications of the complex plane that distinguishes among the values of positive infinity, negative infinity, and other "arguments" of infinity. We augment the complex plane with an ideal set of points on an ideal circle. Each point on this circle corresponds to a different ray in the complex plane emanating from the origin. In this manner we obtain the "ultra extended complex plane." The set of points in the ultra extended complex plane maps to a bowl-shaped subset of the Riemann sphere via a certain projection. We obtain the Riemann stereographic projection as a degenerate limit of a family of projections. Thus, we demonstrate how the infinitely many different directions at infinity degenerate into a single ideal point at infinity that is added to $\mathbb{R}^2$ in order to produce the extended complex plane. The features of this mapping are studied; the introduction of a metric on the ultra extended complex plane is a focal subject of this paper.

1. INTRODUCTION

The stereographic projection provides a one to one mapping between the points of the extended complex plane and the points on the Riemann sphere. Basic text books in complex analysis, e.g. [2, 6], inform us that the stereographic projection is useful for many purposes and serves as an important tool for the concrete visualization of the extended complex plane. The image of the ideal point infinity, the north pole of the Riemann sphere, is treated like the image of any other point in the complex plane. The metric induced by the stereographic projection is well exploited in the theory of meromorphic and entire functions. The fundamental idea is endemic to topology and a useful tool in combinatorial topology (e.g. see [5]). However, the stereographic projection does not distinguish between positive infinity or negative infinity, or among other, different "values" of infinity. It is important both in mathematics and mathematical physics to possess a mean that will distinguish between various "arguments of infinity." Our work addresses this issue by proposing a family of projections in which the stereographic projection becomes a particular, albeit degenerate, case. We study a family of compactifications of the complex plane that takes into account the argument of a point with infinite magnitude. We augment the complex plane $\mathbb{C}$ with an ideal set of points,

$$\text{ID } = \{\infty(\cos\theta,\ \sin\theta) \mid 0 \le \theta < 2\pi\},$$

on an ideal circle. Each point on this circle corresponds to a different ray emanating from the origin. In this manner we obtain the "ultra extended complex plane,"

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which is the union $\mathbb{C}\cup$ID. This set of points in the ultra extended complex plane is mapped onto a subset of the Riemann sphere having the shape of a bowl, essentially the Riemann sphere with a certain cap removed. Thus, we demonstrate how the infinitely many different directions at infinity degenerate into a single ideal point at infinity that is added to $\mathbb{R}^2$ in order to produce the extended complex plane.

Remark 1. The image of the sequence $z_n = (-1)^n n$ converges on the Riemann sphere to the north pole. However, it will not converge in the metric introduced in this paper.

More generally, given a sequence z_n, e^{z_n} always converges on the Riemann sphere with $\mathrm{Re}\{z_n\} \to \infty$. However, with $\mathrm{Re}\{z_n\} \to \infty$, e^{z_n} does not converge in the ultra-extended complex plane unless $e^{i\mathrm{Im}\{z_n\}}$ converges.

The study introduces a metric in a natural manner, the lengthy derivation of which is given in §2. The metric is a focal subject of this paper.

To the best of our knowledge, the properties of the family of compactifications studied in here differ from those of the numerous compactifications in the literature. In particular, we did not encounter a metric such as the one developed in §2. Note that projective geometry, e.g. [7], also provides a mean that distinguishes among the various arguments of infinity.

We note that the Poincare compactification (see [1]) utilizes a projection on two half spheres, as an *intermediate* step, to project an $\mathbb{R}^2$ plane tangent to a sphere onto a perpendicular plane that is also tangent to the sphere. However, neither the bijection nor the metric is computed in this process.

This work should be considered as a prototype that promotes the concept of the ultra extended complex plane and its various compactifications. Indeed, variations of the ideas presented in here seem to benefit our understanding of unbounded convex sets. See [3]. The projection of the ultra extended complex plane onto a parabolic surface is useful for approximation and computation of unbounded functions. See [4].

We now adopt some nomenclature and notation that will be used throughout this paper.

Notation 1.1. *Denote by $Z = (x_1, x_2, x_3)$ a point in the Euclidean space $\mathbb{R}^3$, where x_j satisfy $-\infty < x_j < \infty$, $j = 1, 2, 3$. Denote by $z = (x, y)$ a point in the ultra extended complex plane which is identified with the point $Q = (x, y, 0)$. Let $P = (0,\ 0,\ \gamma)$ be a fixed point on the x_3 coordinate, $0 < \gamma \le 1$. We also put $r^2 = x^2 + y^2$ and $\omega = \gamma^2 + (1 - \gamma^2)r^2$. The word Bowl stands for the following set of points*

$$Bowl = \left\{ \ (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1 \ and \ -1 \le x_3 \le \gamma \right\}$$

The mapping soon to be developed, that matches each point z with a point Z on the Bowl, will be denoted by $G(z)$.

The derivation of the mapping from $\mathbb{C} \cup$ ID to the Bowl is as follows:

If P, Z, and Q lie on the same straight line, then the vectors $\overrightarrow{PZ}$ and $\overrightarrow{PQ}$ are colinear. This is if and only if

$$\overrightarrow{PZ} = t\overrightarrow{PQ}$$

for some real number t.

Since we want our mapping to contain the stereographic projection as a particular case, we require t to be positive.

Thus we have

$$\overrightarrow{PZ} = t\overrightarrow{PQ} \qquad \Leftrightarrow \qquad (x_1,\ x_2,\ x_3-\gamma) = t(x,\ y,\ -\gamma), \tag{1.1}$$

which implies that

$$x_1 = tx, \tag{1.2}$$
$$x_2 = ty, \tag{1.3}$$

and

$$x_3 - \gamma = -t\gamma \qquad \Leftrightarrow \qquad x_3 = \gamma - t\gamma = (1-t)\gamma. \tag{1.4}$$

Since x_1, x_2, and x_3 are points on the unit sphere, we have of course

$$x_1^2 + x_2^2 + x_3^2 = 1. \tag{1.5}$$

We substitute the values of x_1, x_2, and x_3 from equations (1.2), (1.3), and (1.4), respectively, into equation (1.5) to obtain

$$t^2x^2 + t^2y^2 + (1-t)^2\gamma^2 = 1. \tag{1.6}$$

Recall that $r^2 = x^2 + y^2$. It follows from (1.6) that

$$t^2\left(r^2+\gamma^2\right) + \gamma^2\left(1-2t\right) = 1, \tag{1.7}$$

Solving for t we obtain

$$t_{+,-} = \frac{\gamma^2 \pm \sqrt{\gamma^4 - (\gamma^2-1)\left(r^2+\gamma^2\right)}}{\gamma^2+r^2}. \tag{1.8}$$

Because we want $G(z)$ to map to the Bowl, so that $x_3 \le \gamma$, we always choose

$$t = t_+ = \frac{\gamma^2 + \sqrt{\gamma^4 - \gamma^4 + \gamma^2 + (1-\gamma^2)\,r^2}}{\gamma^2+r^2} \tag{1.9}$$

$$= \frac{\gamma^2 + \sqrt{\gamma^2 + (1-\gamma^2)r^2}}{\gamma^2+r^2} \tag{1.10}$$

$$= \frac{\gamma^2+\sqrt{\omega}}{\gamma^2+r^2}. \tag{1.11}$$

We now define our projection $G(z)$.

Definition 1.2. *Let $z = re^{i\theta}$, for $0 \le \theta < 2\pi$. Put*

(1.12)

$$G(z) = \begin{cases} (x_1 = tx, & x_2 = ty, & x_3 = \gamma(1-t) \quad) \ \textit{if} \ \ z \in \mathbb{C} \\ (x_1 = \sqrt{1-\gamma^2}\cos\theta, & x_2 = \sqrt{1-\gamma^2}\sin\theta, & x_3 = \gamma \quad) \ \textit{if} \ \ z = \infty(\cos\theta, \sin\theta) \end{cases}$$

Remark 2. For $\gamma = 1$, we obtain

$$t = \frac{2}{1+r^2},\ x_1 = tx,\ x_2 = ty, \ \text{ and } x_3 = 1-t,$$

the formulas of the stereographic projection. Compare e.g. with [5].

To see that the definition of G given above is natural, first assume $0 < \gamma < 1$. Given a sequence

$$z_n = r_n(\cos\theta_n, \sin\theta_n) = (x_n,\ y_n) \tag{1.13}$$

where $r_n \to \infty$ and $(\cos\theta_n, \sin\theta_n) \to (\cos\theta, \sin\theta)$ as $n \to \infty$, the sequence $Z_n = (x_{1n}, x_{2n}, x_{3n})$ is such that

$$\omega_n = \gamma^2 + (1-\gamma^2)r_n^2 \sim (1-\gamma^2)r_n^2, \tag{1.14}$$

$$\sqrt{\omega_n} \sim \sqrt{1-\gamma^2}\, r_n, \tag{1.15}$$

$$t_n = \frac{\gamma^2 + \sqrt{\omega_n}}{\gamma^2 + r_n^2} \sim \frac{\sqrt{1-\gamma^2}}{r_n}, \tag{1.16}$$

where $\sim$ is the asymptotic equivalence notation. Namely, two functions $f(r)$ and $g(r)$ satisfy

$$f(r) \sim g(r) \text{ as } r \to \infty \text{ iff } \lim_{r\to\infty} \frac{f(r)}{g(r)} = 1.$$

Moreover, we see that

$$\lim_{n\to\infty} t_n = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{3n} = \gamma. \tag{1.17}$$

Furthermore,

$$\begin{aligned} x_{1n} &\sim \tfrac{\sqrt{1-\gamma^2}}{r_n} r_n \cos\theta_n \to \sqrt{1-\gamma^2}\cos\theta, \\ x_{2n} &\sim \tfrac{\sqrt{1-\gamma^2}}{r_n} r_n \sin\theta_n \to \sqrt{1-\gamma^2}\sin\theta, \end{aligned} \tag{1.18}$$

as $n \to \infty$.

For $\gamma = 1$, the sequence Z_n is such that

$$\omega_n = \gamma^2 + (1-\gamma^2)r_n^2 = 1, \tag{1.19}$$

$$t_n = \frac{\gamma^2 + \sqrt{\omega_n}}{\gamma^2 + r_n^2} \sim \frac{2}{1 + r_n^2}. \tag{1.20}$$

Furthermore,

$$\lim_{n\to\infty} t_n = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{3n} = 1, \tag{1.21}$$

$$\begin{aligned} x_{1n} &\to \tfrac{2}{1+r_n^2} r_n \cos\theta_n \to 0, \\ x_{2n} &\to \tfrac{2}{1+r_n^2} r_n \sin\theta_n \to 0, \end{aligned} \tag{1.22}$$

as $n \to \infty$.

We now turn to some properties of the function G. To study these properties, we employ the following notation.

Notation 1.3. *Denote by $\hat{Z} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ a point in the Euclidean space $\mathbb{R}^3$, where $\hat{x}_j$ satisfy $-\infty < \hat{x}_j < \infty$, $j = 1, 2, 3$. Denote by $\hat{z} = (\hat{x}, \hat{y})$ a point in the ultra extended complex plane which is identified with the point $\hat{Q} = (\hat{x}, \hat{y}, 0)$. We also put*

$$\hat{r}^2 = \hat{x}^2 + \hat{y}^2, \ \hat{\omega} = \gamma^2 + (1-\gamma^2)\hat{r}^2, \ \textit{and } \hat{t} = \frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}.$$

The following theorem establishes an important property of G.

Theorem 1.4. *G is a bijection from the ultra extended complex plane to the Bowl.*

Proof. First we show that G is one-to-one. Let $z, \hat{z} \in \mathbb{C} \cup \mathrm{ID}$ and $Z = G(z)$, $\hat{Z} = G(\hat{z})$. Then Z and $\hat{Z}$ lie on the Bowl. We shall examine the one-to-one-ness of G in two cases.

Case 1. $x_3, \hat{x}_3 < \gamma$. From Definition 1.2 we have $x_3, \hat{x}_3 < \gamma$ if and only if $z, \hat{z} \in \mathbb{C}$.

In the sequel we will need the range of $t = \frac{\gamma^2+\sqrt{\omega}}{\gamma^2+r^2}$.

It is easy to verify that t is a monotone decreasing function of r^2 since

$$\frac{\partial t}{\partial(r^2)} = \frac{-t^2}{2\sqrt{\omega}} < 0,\ 0 \le r^2 < \infty. \tag{1.23}$$

This tells us that

$$1 + \frac{1}{\gamma} > t > 0. \tag{1.24}$$

Evidently $t = 0$ iff $r = \infty$ iff $x_3 = \gamma$. In the case of $z, \hat{z} \in \mathbb{C}$, $t, \hat{t} \neq 0$.

For $Z = \hat{Z}$, we know that $x_1 = \hat{x}_1$, $x_2 = \hat{x}_2$, and $x_3 = \hat{x}_3$. Expanding $x_3 = \hat{x}_3$, we see that

$$x_3 = \gamma(1-t) = \hat{x}_3 = \gamma(1-\hat{t}) \tag{1.25}$$

and deduce that $t = \hat{t}$.

Examining x_1 and $\hat{x}_1$, we have the relation

$$x_1 = \hat{x}_1 \quad \Leftrightarrow \quad tx = \hat{t}\hat{x} \quad \Leftrightarrow \quad x = \hat{x}. \tag{1.26}$$

Likewise,

$$x_2 = \hat{x}_2 \quad \Leftrightarrow \quad y = \hat{y}. \tag{1.27}$$

Therefore $Z = \hat{Z}$ iff $z = \hat{z}$, and G is one-to-one.

Case 2. $x_3 = \hat{x}_3 = \gamma$. From definition (1.2), $x_3 = \hat{x}_3 = \gamma$ iff $z, \hat{z} \in$ ID.

Examining z and $\hat{z}$, we see that $z = \hat{z}$ iff

$$\left(\sqrt{1-\gamma^2}\cos\theta,\ \sqrt{1-\gamma^2}\sin\theta,\ \gamma\right) = \left(\sqrt{1-\gamma^2}\cos\hat{\theta},\ \sqrt{1-\gamma^2}\sin\hat{\theta},\ \gamma\right).$$

This is true iff $\cos\theta = \cos\hat{\theta}$ and $\sin\theta = \sin\hat{\theta}$, which implies that $\theta = \hat{\theta}$ for $0 \le \theta, \hat{\theta} < 2\pi$ (by definition). The equality of θ and $\hat{\theta}$ implies that

$$\infty(\cos\theta, \sin\theta) = \infty(\cos\theta, \sin\theta),$$

which gives us our desired result, that G is one-to-one.

Remark 3. Obviously, a *Case* 3 where $x_3 < \gamma$ and $\hat{x}_3 = \gamma$ cannot occur.

Therefore, G is one-to-one for all $z, \hat{z} \in \mathbb{C} \cup$ ID.

Next we show that G is onto. For any $Z = (x_1, x_2, x_3)$ on the Bowl, either $x_3 < \gamma$ or $x_3 = \gamma$.

Case 1.

$$x_3 < \gamma \Leftrightarrow (1-t)\gamma < \gamma \Leftrightarrow 1 - t < 1 \Leftrightarrow -t < 0 \Leftrightarrow t > 0 \tag{1.28}$$

Solving for t yields

$$x_3 = (1-t)\gamma \Leftrightarrow \frac{x_3}{\gamma} = 1 - t \Leftrightarrow t = 1 - \frac{x_3}{\gamma}. \tag{1.29}$$

Substituting (1.29) into $x_1 = tx$ and $x_2 = ty$, we obtain

$$x_1 = tx \quad \Leftrightarrow \quad x = \frac{x_1}{t} = \frac{x_1}{1 - \frac{x_3}{\gamma}}, \tag{1.30}$$

$$x_2 = ty \quad \Leftrightarrow \quad y = \frac{x_2}{t} = \frac{x_2}{1 - \frac{x_3}{\gamma}}. \tag{1.31}$$

Therefore,

$$G\left(\frac{x_1}{1-\frac{x_3}{\gamma}},\ \frac{x_2}{1-\frac{x_3}{\gamma}}\right) = Z. \tag{1.32}$$

Case 2. $x_3 = \gamma$.

Now we match the following $z \in \mathrm{ID}$ to Z:

$(x_1,\ x_2)$ lies on a circle with radius $\sqrt{1-\gamma^2}$, hence there exists a unique θ, $0 \le \theta < 2\pi$, such that

$$x_1 \quad = \quad \sqrt{1-\gamma^2}\cos\theta \quad \text{ and } \quad x_2 \quad = \quad \sqrt{1-\gamma^2}\sin\theta.$$

Thus, $z = \infty(\cos\theta,\ \sin\theta)$, and, therefore, G is onto and one-to-one, a bijection. □

With an eye to proving completeness of $\mathbb{C}\cup\mathrm{ID}$ in the metric to be defined in the next section we add the following definition.

Definition 1.5. *We say that the sequence* $r_n(\cos\theta_n, \sin\theta_n)$ *converges to a point* $\infty(\cos\theta,\ \sin\theta)$ *if*

$$\lim_{r\to\infty} r_n = \infty$$

and

$$\lim_{n\to\infty}\cos\theta_n = \cos\theta, \quad \lim_{n\to\infty}\sin\theta_n = \sin\theta, \quad \textit{for some } \theta,\ 0 \le \theta < 2\pi$$

2. An Induced Metric

In this section we derive a metric $\chi(z,\ \hat{z})$ for the ultra extended complex plane. In the sequel we denote by $\|G(z) - G(\hat{z})\|$ the Euclidean distance between two points.

Theorem 2.1. *The ultra extended complex plane is a complete metric space with respect to the chordal metric* χ *defined in (2.1) below as the Euclidean distance* $\|G(z) - G(\hat{z})\|$.

$$\chi(z,\hat{z}) \equiv \|G(z) - G(\hat{z})\| = \sqrt{(x_1-\hat{x}_1)^2 + (x_2-\hat{x}_2)^2 + (x_3-\hat{x}_3)^2}. \tag{2.1}$$

Specifically, the square of the metric χ^2 *is given by*

$$\chi^2(z,\hat{z}) = F(D^2 - \Delta) \tag{2.2}$$

for $z = (x,y)$, $\hat{z} = (\hat{x},\hat{y})$, *where* F *is a dilation factor*

$$F = \frac{(\gamma^2+\sqrt{\omega})(\gamma^2+\sqrt{\hat{\omega}})}{(\gamma^2+r^2)(\gamma^2+\hat{r}^2)}, \tag{2.3}$$

D^2 *is the square of the Euclidean distance between* z *and* $\hat{z}$

$$D^2 = (x-\hat{x})^2 + (y-\hat{y})^2, \tag{2.4}$$

and Δ *is a "positive definite" measure of the distance between* r^2 *and* $\hat{r}^2$. *Specifically,* Δ *is given by a non-negative function of all the variables* $\gamma^2 \le 1$, r^2 *and* $\hat{r}^2$. *Namely,*

(2.5)
$$\Delta = \frac{(1-\gamma^2)(r^2-\hat{r}^2)^2}{(\sqrt{\omega}+\sqrt{\hat{\omega}})(\gamma^2+\sqrt{\omega})(\gamma^2+\sqrt{\hat{\omega}})}\left[\gamma^2 + \frac{(1-\gamma^2)r^2\hat{r}^2 + \gamma^4(1+\gamma^2) + \gamma^2(r^2+\hat{r}^2)}{(\gamma^2+r^2)\sqrt{\hat{\omega}} + (\gamma^2+\hat{r}^2)\sqrt{\omega}}\right].$$

Moreover, $\Delta \geq 0$ *for* $\gamma^2 < 1$ *and* $\Delta = 0$ *iff* $\gamma^2 = 1$ *or* $r^2 = \hat{r}^2$.

For $z = \infty(\cos\theta, \sin\theta)$, $\hat{z} = (\hat{x}, \hat{y})$,

$$\chi^2(z, \hat{z}) = 2\gamma^2 \frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2} + 2(1-\gamma^2) - 2\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\sqrt{1-\gamma^2}(\hat{x}\cos\theta + \hat{y}\sin\theta) \tag{2.6}$$

and for $z = \infty(\cos\theta, \sin\theta)$, $\hat{z} = \infty(\cos\hat{\theta}, \sin\hat{\theta})$,

$$\chi^2(z, \hat{z}) = 4(1-\gamma^2)\sin^2\left(\frac{\theta - \hat{\theta}}{2}\right) = \left(2\sqrt{1-\gamma^2}\left|\sin\left(\frac{\theta-\hat{\theta}}{2}\right)\right|\right)^2 \tag{2.7}$$

Proof. To show that $\chi(z, \hat{z})$ is a distance function, χ must satisfy the following conditions:

(i) $\chi(z, \hat{z}) \geq 0$, with equality only for $z = \hat{z}$.

(ii) $\chi(z, \hat{z}) = \chi(\hat{z}, z)$.

(iii) $\chi(z, \hat{z}) \leq \chi(z, \tilde{z}) + \chi(\tilde{z}, \hat{z})$.

Recall that χ is defined as the Euclidean distance function between $G(z)$ and $G(\hat{z})$ in $\mathbb{R}^3$.

Therefore, $\|G(z) - G(\hat{z})\| \geq 0$, with equality for $G(z) = G(\hat{z})$. Because G is one-to-one, $G(z) = G(\hat{z})$ iff $z = \hat{z}$. This proves (i).

Next,

$$\chi(z, \hat{z}) = \|G(z) - G(\hat{z})\| = \|G(\hat{z}) - G(z)\| = \chi(\hat{z}, z). \tag{2.8}$$

This proves (ii).

Let $Z = G(z)$, $\hat{Z} = G(\hat{z})$, and $\tilde{Z} = G(\tilde{z})$ for three distinct points $z, \hat{z}, \tilde{z} \in \mathbb{C} \cup \mathrm{ID}$. The triangle inequality, (iii), holds because $\triangle Z\hat{Z}\tilde{Z}$ is a triangle in $\mathbb{E}^3$.

Notice that properties (i) and (ii) also follow easily from the explicit formulas (2.2) to (2.7). It is also possible to obtain property (iii) directly from (2.2) to (2.7) through a laborious calculation.

Now we turn to the lengthy proof that the expression of the chordal metric χ is indeed as described in Theorem 2.1.

To derive χ, we turn to the definition of G, namely

$$\|G(z) - G(\hat{z})\|^2 = (x_1 - \hat{x_1})^2 + (y_1 - \hat{y_1})^2 + (z_1 - \hat{z_1})^2. \tag{2.9}$$

For $Z = \hat{Z}$, $\|Z - \hat{Z}\|^2 = 0$. Expanding the squared terms in equation (2.9) gives us

$$\|Z - \hat{Z}\|^2 = x_1^2 + x_2^2 + x_3^2 + \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 - 2x_1\hat{x}_1 - 2x_2\hat{x}_2 - 2x_3\hat{x}_3. \tag{2.10}$$

Because x_1, x_2, x_3 and $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ are coordinates on the unit sphere, we have

$$x_1^2 + x_2^2 + x_3^2 = \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = 1. \tag{2.11}$$

Substituting the expression given in equation (2.11) into equation (2.10) leaves us with

$$\|Z-\hat{Z}\|^2 = 1+1-2\left[x_1\hat{x}_1 + x_2\hat{x}_2 + x_3\hat{x}_3\right]. \tag{2.12}$$

Expanding x_1, x_2, x_3, $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$ given by Definition 1.2 gives us

$$\begin{aligned} &\|Z-\hat{Z}\|^2 \\ &= 2-2\left[\left(\frac{\gamma^2+\sqrt{\omega}}{\gamma^2+r^2}\right)\left(\frac{\gamma^2+\sqrt{\hat{\omega}}}{\gamma^2+\hat{r}^2}\right)(x\hat{x}+y\hat{y}+\gamma^2)\right. \\ &\qquad \left. +\gamma^2-\gamma^2\left(\frac{\gamma^2+\sqrt{\omega}}{\gamma^2+r^2}\right)-\gamma^2\left(\frac{\gamma^2+\sqrt{\hat{\omega}}}{\gamma^2+\hat{r}^2}\right)\right]. \end{aligned} \tag{2.13}$$

Expanding D^2 yields

$$D^2 = x^2+\hat{x}^2+y^2+\hat{y}^2-2x\hat{x}-2y\hat{y}. \tag{2.14}$$

Substituting $r^2 = x^2+y^2$ and $\hat{r}^2 = \hat{x}^2+\hat{y}^2$ into equation (2.14) shows us that

$$D^2-r^2-\hat{r}^2 = -2x\hat{x}-2y\hat{y}. \tag{2.15}$$

Next we substitute the expression given in equation (2.15) into equation (2.13) to obtain

$$\begin{aligned} &\|Z-\hat{Z}\|^2 \\ &= \left(\frac{\gamma^2+\sqrt{\omega}}{\gamma^2+r^2}\right)\left(\frac{\gamma^2+\sqrt{\hat{\omega}}}{\gamma^2+\hat{r}^2}\right)(D^2-r^2-\hat{r}^2)+2-2\gamma^2 \\ &\quad -2\gamma^2\left(\frac{\gamma^2+\sqrt{\omega}}{\gamma^2+r^2}\right)\left(\frac{\gamma^2+\sqrt{\hat{\omega}}}{\gamma^2+\hat{r}^2}\right)+2\gamma^2\left(\frac{\gamma^2+\sqrt{\omega}}{\gamma^2+r^2}\right)+2\gamma^2\left(\frac{\gamma^2+\sqrt{\hat{\omega}}}{\gamma^2+\hat{r}^2}\right). \end{aligned} \tag{2.16}$$

Substituting $p = \gamma^2+r^2$, $\hat{p} = \gamma^2+\hat{r}^2$ and $q = \gamma^2+\sqrt{\omega}$, $\hat{q} = \gamma^2+\sqrt{\hat{\omega}}$ into equation (2.16), we are left with

$$\begin{aligned} \|Z-\hat{Z}\|^2 &= \left(\frac{q}{p}\right)\left(\frac{\hat{q}}{\hat{p}}\right)(D^2-r^2-\hat{r}^2)+2-2\gamma^2 \\ &\quad -2\gamma^2\left(\frac{q}{p}\right)\left(\frac{\hat{q}}{\hat{p}}\right)+2\gamma^2\left(\frac{q}{p}\right)+2\gamma^2\left(\frac{\hat{q}}{\hat{p}}\right). \end{aligned} \tag{2.17}$$

Setting all terms in equation (2.17) to the denominator $(p\hat{p})$, we obtain

$$\|Z-\hat{Z}\|^2 = \frac{q\hat{q}}{p\hat{p}}(D^2-r^2-\hat{r}^2)+2(1-\gamma^2)\frac{p\hat{p}}{p\hat{p}}-2\gamma^2\frac{q\hat{q}}{p\hat{p}}+2\gamma^2\frac{q\hat{p}}{p\hat{p}}+2\gamma^2\frac{\hat{q}p}{p\hat{p}}. \tag{2.18}$$

Let

$$\|Z-\hat{Z}\|^2 = S_1+S_2+S_3, \tag{2.19}$$

where

$$S_1 = \frac{(D^2-r^2-\hat{r}^2)(q\hat{q})}{p\hat{p}}, \tag{2.20}$$

$$S_2 = \frac{2(1-\gamma^2)(p\hat{p})}{p\hat{p}}, \tag{2.21}$$

and

$$S_3 = \frac{2\gamma^2(q\hat{p})+2\gamma^2(\hat{q}p)-2\gamma^2(q\hat{q})}{p\hat{p}}. \tag{2.22}$$

Expanding $q\hat{q}$, $q\hat{p}$, $\hat{q}p$ in S_3 leads to

$$\begin{aligned} S_3 &= \frac{2\gamma^2(\gamma^4+\gamma^2\hat{r}^2+\gamma^2\sqrt{\omega}+\hat{r}^2\sqrt{\omega})}{p\hat{p}} \\ &\quad+\frac{2\gamma^2(\gamma^4+\gamma^2 r^2+\gamma^2\sqrt{\hat{\omega}}+r^2\sqrt{\hat{\omega}})}{p\hat{p}} \\ &\quad-\frac{2\gamma^2(\gamma^4+\gamma^2\sqrt{\hat{\omega}}+\gamma^2\sqrt{\omega}+\sqrt{\omega}\sqrt{\hat{\omega}})}{p\hat{p}}. \end{aligned} \tag{2.23}$$

Cancelling like terms in equation (2.23), we find that

$$S_3 = \frac{2\gamma^6+2\gamma^4 r^2+2\gamma^4\hat{r}^2+2\gamma^2 r^2\sqrt{\hat{\omega}}+2\gamma^2\hat{r}^2\sqrt{\omega}-2\gamma^2\sqrt{\omega}\sqrt{\hat{\omega}}}{p\hat{p}}. \tag{2.24}$$

In S_1 we split $(D-r^2-\hat{r}^2)$ and expand $q\hat{q}$ to obtain

$$\begin{aligned} S_1 &= \frac{q\hat{q}}{p\hat{p}}D^2 \\ &\quad -r^2\frac{(\gamma^4+\gamma^2\sqrt{\omega}+\gamma^2\sqrt{\hat{\omega}}+\sqrt{\omega}+\sqrt{\hat{\omega}})}{p\hat{p}} \\ &\quad -\hat{r}^2\frac{(\gamma^4+\gamma^2\sqrt{\omega}+\gamma^2\sqrt{\hat{\omega}}+\sqrt{\omega}+\sqrt{\hat{\omega}})}{p\hat{p}} \end{aligned} \tag{2.25}$$

$$\begin{aligned} &= \frac{q\hat{q}}{p\hat{p}}D^2 \\ &\quad +\frac{-\gamma^4\hat{r}^2-\gamma^2\hat{r}^2\sqrt{\omega}-\gamma^4 r^2-\gamma^2 r^2\sqrt{\hat{\omega}}-\gamma^2\hat{r}^2\sqrt{\hat{\omega}}}{p\hat{p}} \\ &\quad +\frac{-\hat{r}^2\sqrt{\omega}\sqrt{\hat{\omega}}-\gamma^2 r^2\sqrt{\omega}-r^2\sqrt{\omega}\sqrt{\hat{\omega}}}{p\hat{p}}. \end{aligned} \tag{2.26}$$

Adding together S_1 and S_2 yields

(2.27)
$$\begin{aligned} S_1+S_3 &= \frac{q\hat{q}}{p\hat{p}}D^2 \\ &+\frac{2\gamma^6+2\gamma^4r^2+2\gamma^4\hat{r}^2+2\gamma^2r^2\sqrt{\hat{\omega}}+2\gamma^2\hat{r}^2\sqrt{\omega}-2\gamma^2\sqrt{\omega}\sqrt{\hat{\omega}}}{p\hat{p}} \\ &+\frac{-\gamma^4\hat{r}^2-\gamma^2\hat{r}^2\sqrt{\omega}-\gamma^4r^2-\gamma^2r^2\sqrt{\hat{\omega}}-\gamma^2\hat{r}^2\sqrt{\hat{\omega}}}{p\hat{p}} \\ &+\frac{-\hat{r}^2\sqrt{\omega}\sqrt{\hat{\omega}}-\gamma^2r^2\sqrt{\omega}-r^2\sqrt{\omega}\sqrt{\hat{\omega}}}{p\hat{p}}. \end{aligned}$$

Cancelling like terms in equation (2.27) shows us that

(2.28)
$$\begin{aligned} S_1+S_3 &= \frac{q\hat{q}}{p\hat{p}}D^2 \\ &+\frac{2\gamma^6-2\gamma^2\sqrt{\omega}\sqrt{\hat{\omega}}-r^2\sqrt{\omega}\sqrt{\hat{\omega}}-\hat{r}^2\sqrt{\omega}\sqrt{\hat{\omega}}}{p\hat{p}} \\ &+\frac{-\gamma^2r^2\sqrt{\omega}-\gamma^2\hat{r}^2\sqrt{\hat{\omega}}+\gamma^4\hat{r}^2+\gamma^2\hat{r}^2\sqrt{\omega}+\gamma^4r^2+\gamma^2r^2\sqrt{\hat{\omega}}}{p\hat{p}}. \end{aligned}$$

Now we split $S_1+S_2+S_3$ into the factor containing D^2 and the rest. Factoring $\frac{q\hat{q}}{p\hat{p}}$ from $\|Z-\hat{Z}\|^2 = S_1+S_2+S_3$ yields

(2.29)
$$\|Z-\hat{Z}\|^2 = S_1+S_2+S_3 = \frac{q\hat{q}}{p\hat{p}}\left\{D^2-H\right\} = F\left\{D^2-H\right\},$$

where

$$\begin{aligned} H &= \frac{-2\gamma^6+2\gamma^2\sqrt{\omega}\sqrt{\hat{\omega}}+r^2\sqrt{\omega}\sqrt{\hat{\omega}}+\hat{r}^2\sqrt{\omega}\sqrt{\hat{\omega}}}{q\hat{q}} \\ &+\frac{\gamma^2r^2\sqrt{\omega}+\gamma^2\hat{r}^2\sqrt{\hat{\omega}}-\gamma^4\hat{r}^2-\gamma^2\hat{r}^2\sqrt{\omega}-\gamma^4r^2-\gamma^2r^2\sqrt{\hat{\omega}}}{q\hat{q}} \\ &+\frac{-2(1-\gamma^2)(p\hat{p})}{q\hat{q}}. \end{aligned} \tag{2.30}$$

Multiplying the numerators and denominators of H by $(\sqrt{\omega}+\sqrt{\hat{\omega}})$, we obtain

$$\begin{aligned} H &= \frac{(\sqrt{\omega}+\sqrt{\hat{\omega}})}{(\sqrt{\omega}+\sqrt{\hat{\omega}})}\frac{\left(-2\gamma^6+2\gamma^2\sqrt{\omega}\sqrt{\hat{\omega}}+r^2\sqrt{\omega}\sqrt{\hat{\omega}}+\hat{r}^2\sqrt{\omega}\sqrt{\hat{\omega}}\right)}{q\hat{q}} \\ &+\frac{(\sqrt{\omega}+\sqrt{\hat{\omega}})}{(\sqrt{\omega}+\sqrt{\hat{\omega}})}\frac{\left(\gamma^2r^2\sqrt{\omega}+\gamma^2\hat{r}^2\sqrt{\hat{\omega}}\right)}{q\hat{q}} \\ &+\frac{(\sqrt{\omega}+\sqrt{\hat{\omega}})}{(\sqrt{\omega}+\sqrt{\hat{\omega}})}\frac{\left(-\gamma^4\hat{r}^2-\gamma^2\hat{r}^2\sqrt{\omega}-\gamma^4r^2-\gamma^2r^2\sqrt{\hat{\omega}}\right)}{q\hat{q}} \\ &+\frac{(\sqrt{\omega}+\sqrt{\hat{\omega}})}{(\sqrt{\omega}+\sqrt{\hat{\omega}})}\frac{\left(-2(1-\gamma^2)p\hat{p}\right)}{q\hat{q}}. \end{aligned} \tag{2.31}$$

Consider H as a quotient of two quantities, $Numerator(H)$ and $Denominator(H)$, where

(2.32)
$$Denominator(H) = (\sqrt{\omega}+\sqrt{\hat{\omega}})(q\hat{q})$$
and

(2.33)
$$Numerator(H) = H_1 + H_2,$$
where

$$H_1 = (\sqrt{\omega}+\sqrt{\hat{\omega}})\left[2\gamma^2\sqrt{\omega}\sqrt{\hat{\omega}} + r^2\sqrt{\omega}\sqrt{\hat{\omega}} + \hat{r}^2\sqrt{\omega}\sqrt{\hat{\omega}}\right] + (\sqrt{\omega}+\sqrt{\hat{\omega}})\left[\gamma^2 r^2\sqrt{\omega} + \gamma^2\hat{r}^2\sqrt{\hat{\omega}} - \gamma^2\hat{r}^2\sqrt{\omega} - \gamma^2 r^2\sqrt{\hat{\omega}}\right], \tag{2.34}$$

$$H_2 = -(\sqrt{\omega}+\sqrt{\hat{\omega}})\left[2\gamma^6 + \gamma^4\hat{r}^2 + \gamma^4 r^2 + 2(1-\gamma^2)p\hat{p}\right]. \tag{2.35}$$

Expanding $(\sqrt{\omega}+\sqrt{\hat{\omega}})$ in H_1 gives us

(2.36)
$$\begin{aligned} H_1 &= 2\gamma^2\sqrt{\hat{\omega}}\omega + r^2\sqrt{\hat{\omega}}\omega + \hat{r}^2\sqrt{\hat{\omega}}\omega + \gamma^2 r^2\omega + \gamma^2\hat{r}^2\sqrt{\omega}\sqrt{\hat{\omega}} - \gamma^2\hat{r}^2\omega - \gamma^2 r^2\sqrt{\omega}\sqrt{\hat{\omega}} \\ &\quad + 2\gamma^2\sqrt{\omega}\hat{\omega} + r^2\sqrt{\omega}\hat{\omega} + \hat{r}^2\sqrt{\omega}\hat{\omega} + \gamma^2\hat{r}^2\hat{\omega} - \gamma^2\hat{r}^2\sqrt{\omega}\sqrt{\hat{\omega}} - \gamma^2 r^2\hat{\omega} + \gamma^2 r^2\sqrt{\omega}\sqrt{\hat{\omega}}. \end{aligned}$$

Substituting $\omega = \gamma^2 + (1-\gamma^2)r^2$ and $\hat{\omega} = \gamma^2 + (1-\gamma^2)\hat{r}^2$ into equation (2.36), we get

(2.37)
$$\begin{aligned} H_1 &= 2\gamma^2\sqrt{\hat{\omega}}(\gamma^2 + (1-\gamma^2)r^2) + r^2\sqrt{\hat{\omega}}(\gamma^2 + (1-\gamma^2)r^2) + \hat{r}^2\sqrt{\hat{\omega}}(\gamma^2 + (1-\gamma^2)r^2) \\ &\quad + \gamma^2 r^2(\gamma^2 + (1-\gamma^2)r^2) + \gamma^2\hat{r}^2\sqrt{\omega}\sqrt{\hat{\omega}} - \gamma^2\hat{r}^2(\gamma^2 + (1-\gamma^2)r^2) - \gamma^2 r^2\sqrt{\omega}\sqrt{\hat{\omega}} \\ &\quad + 2\gamma^2\sqrt{\omega}(\gamma^2 + (1-\gamma^2)\hat{r}^2) + r^2\sqrt{\omega}(\gamma^2 + (1-\gamma^2)\hat{r}^2) + \hat{r}^2\sqrt{\omega}(\gamma^2 + (1-\gamma^2)\hat{r}^2) \\ &\quad + \gamma^2\hat{r}^2(\gamma^2 + (1-\gamma^2)\hat{r}^2) + \gamma^2 r^2\sqrt{\omega}\sqrt{\hat{\omega}} - \gamma^2 r^2(\gamma^2 + (1-\gamma^2)\hat{r}^2) - \gamma^2\hat{r}^2\sqrt{\omega}\sqrt{\hat{\omega}}. \end{aligned}$$

Factoring $(1-\gamma^2)$ from equation (2.37) yields

$$H_1 = I_1 + I_2 \tag{2.38}$$
where

$$I_1 = (1-\gamma^2)\left[2\gamma^2 r^2\sqrt{\hat{\omega}} + r^4\sqrt{\hat{\omega}} + r^2\hat{r}^2\sqrt{\hat{\omega}} + r^4\gamma^2 - r^2\hat{r}^2\gamma^2\right] + (1-\gamma^2)\left[2\gamma^2\hat{r}^2\sqrt{\omega} + \hat{r}^4\sqrt{\omega} + r^2\hat{r}^2\sqrt{\omega} + \hat{r}^4\gamma^2 - r^2\hat{r}^2\gamma^2\right] \tag{2.39}$$

and

$$I_2 = \left[2\gamma^4\sqrt{\omega} + \gamma^2\hat{r}^2\sqrt{\omega} + \gamma^2 r^2\sqrt{\omega}\right] + \left[2\gamma^4\sqrt{\hat{\omega}} + \gamma^2\hat{r}^2\sqrt{\hat{\omega}} + \gamma^2 r^2\sqrt{\hat{\omega}}\right] + \underbrace{\gamma^4 r^2 - \gamma^4 r^2 + \gamma^4\hat{r}^2 - \gamma^4\hat{r}^2}_{0}. \tag{2.40}$$

Expanding $(\sqrt{\omega}+\sqrt{\hat\omega})$ in H_2 gives us

$$H_2 = -\left[2\gamma^6\sqrt{\omega}+\gamma^4\hat r^2\sqrt{\omega}+\gamma^4 r^2\sqrt{\omega}+2\sqrt{\omega}(1-\gamma^2)p\hat p\right] - \left[2\gamma^6\sqrt{\hat\omega}+\gamma^4\hat r^2\sqrt{\hat\omega}+\gamma^4 r^2\sqrt{\hat\omega}+2\sqrt{\hat\omega}(1-\gamma^2)p\hat p\right] \tag{2.41}$$

$$= -\gamma^2 I_2 + I_3, \tag{2.42}$$

where

$$I_3 = (1-\gamma^2)\left[-2\sqrt{\omega}p\hat p - 2\sqrt{\hat\omega}p\hat p\right]. \tag{2.43}$$

Therefore

$$H_1 + H_2 = Numerator(H) = I_1 + (1-\gamma^2)I_2 + I_3. \tag{2.44}$$

Collecting terms in $I_1 + (1-\gamma^2)I_2$ yields

$$\begin{aligned} &Numerator(H) \\ &= (1-\gamma^2)\left[3\gamma^2 r^2\sqrt{\hat\omega}+3\gamma^2\hat r^2\sqrt{\omega}+r^4\sqrt{\hat\omega}+\hat r^4\sqrt{\omega}+r^2\hat r^2\sqrt{\hat\omega}+r^2\hat r^2\sqrt{\omega}\right] \\ &\quad + (1-\gamma^2)\left[-2\gamma^2r^2\hat r^2+r^4\gamma^2+\hat r^4\gamma^2+\gamma^2\hat r^2\sqrt{\hat\omega}+\gamma^2r^2\sqrt{\omega}+2\gamma^4\sqrt{\hat\omega}+2\gamma^4\sqrt{\omega}\right] \\ &\quad + I_3. \end{aligned} \tag{2.45}$$

Expanding $p\hat p$ in I_3,

$$I_3 = (1-\gamma^2)\left[-2\sqrt{\omega}(\gamma^4+\gamma^2\hat r^2+\gamma^2r^2+r^2\hat r^2) - 2\sqrt{\hat\omega}(\gamma^4+\gamma^2\hat r^2+\gamma^2r^2+r^2\hat r^2)\right]. \tag{2.46}$$

We substitute I_3 from equation (2.46) into equation (2.45) and cancel like terms to obtain

$$\begin{aligned} &Numerator(H) \\ &= (1-\gamma^2)\left[r^4\sqrt{\hat\omega}+\hat r^4\sqrt{\omega}+r^4\gamma^2+\hat r^4\gamma^2-2\gamma^2r^2\hat r^2+\gamma^2r^2\sqrt{\hat\omega}+\gamma^2\hat r^2\sqrt{\omega}\right] \\ &\quad + (1-\gamma^2)\left[-\gamma^2r^2\sqrt{\omega}-\gamma^2\hat r^2\sqrt{\hat\omega}-r^2\hat r^2\sqrt{\omega}-r^2\hat r^2\sqrt{\hat\omega}\right]. \end{aligned} \tag{2.47}$$

Factoring $(r^2-\hat r^2)$ from equation (2.47), we are left with

$$\begin{aligned} &Numerator(H) \\ &= (1-\gamma^2)\left[(r^2-\hat r^2)^2\gamma^2+(r^2-\hat r^2)(\sqrt{\hat\omega}-\sqrt{\omega})\gamma^2+(r^2-\hat r^2)(r^2\sqrt{\hat\omega}-\hat r^2\sqrt{\omega})\right]. \end{aligned} \tag{2.48}$$

Reintroducing $Denominator(H)$ gives us

$$H = (1-\gamma^2)\frac{(r^2-\hat r^2)\left[\gamma^2(r^2-\hat r^2)+\gamma^2(\sqrt{\hat\omega}-\sqrt{\omega})+(r^2\sqrt{\hat\omega}-\hat r^2\sqrt{\omega})\right]}{(\sqrt{\omega}+\sqrt{\hat\omega})(q\hat q)}. \tag{2.49}$$

Substituting $q=\gamma^2+\sqrt{\omega}$ and $\hat q=\gamma^2+\sqrt{\hat\omega}$ into equation (2.49) yields

$$H = (1-\gamma^2)\frac{(r^2-\hat r^2)\left[\gamma^2(r^2-\hat r^2)+\gamma^2(\sqrt{\hat\omega}-\sqrt{\omega})+(r^2\sqrt{\hat\omega}-\hat r^2\sqrt{\omega})\right]}{(\sqrt{\omega}+\sqrt{\hat\omega})(\gamma^2+\sqrt{\omega})(\gamma^2+\sqrt{\hat\omega})}. \tag{2.50}$$

Splitting H into two parts, we are left with

$$H = \frac{(1-\gamma^2)(r^2-\hat{r}^2)}{(\sqrt{\omega}+\sqrt{\hat{\omega}})(\gamma^2+\sqrt{\omega})(\gamma^2+\sqrt{\hat{\omega}})}\psi, \tag{2.51}$$

where

$$\psi = M_1 + M_2, \tag{2.52}$$

$$M_1 = \gamma^2(r^2-\hat{r}^2), \tag{2.53}$$

and

$$M_2 = \gamma^2(\sqrt{\hat{\omega}}-\sqrt{\omega}) + (r^2\sqrt{\hat{\omega}} - \hat{r}^2\sqrt{\omega}). \tag{2.54}$$

Notice that M_2 can also be written as

$$M_2 = (\gamma^2+r^2)\sqrt{\hat{\omega}} - (\gamma^2+\hat{r}^2)\sqrt{\omega}. \tag{2.55}$$

Let

$$\bar{M}_2 = (\gamma^2+r^2)\sqrt{\hat{\omega}} + (\gamma^2+\hat{r}^2)\sqrt{\omega}. \tag{2.56}$$

Multiplying M_2 by $\frac{\bar{M}_2}{\bar{M}_2}$ gives us

$$M_2 = \frac{M_2\bar{M}_2}{\bar{M}_2}. \tag{2.57}$$

Expanding $M_2\bar{M}_2$, we obtain

$$M_2\bar{M}_2 = (\gamma^2+r^2)^2\hat{\omega} + (\gamma^2+\hat{r}^2)^2\omega \tag{2.58}$$

$$= N_1 + N_2, \tag{2.59}$$

where

$$N_1 = (\gamma^2+r^2)^2(\gamma^2+(1-\gamma^2)\hat{r}^2) \tag{2.60}$$

and

$$N_2 = -(\gamma^2+\hat{r}^2)^2(\gamma^2+(1-\gamma^2)r^2). \tag{2.61}$$

After substituting $(\gamma^2+r^2)^2 = (\gamma^4+2\gamma^2r^2+r^4)$ into the above expression for N_1 and $(\gamma^2+\hat{r}^2)^2 = (\gamma^4+2\gamma^2\hat{r}^2+\hat{r}^4)$ into the above expression for N_2, we have

$$N_1 = (\gamma^2+r^2)^2(\gamma^2+(1-\gamma^2)\hat{r}^2) \tag{2.62}$$

and

$$N_2 = -(\gamma^2+\hat{r}^2)^2(\gamma^2+(1-\gamma^2)r^2). \tag{2.63}$$

Expanding N_1 and N_2 as given in equations (2.62) and (2.63) yields

(2.64)

$$N_1 = \gamma^6 + \gamma^4(1-\gamma^2)\hat{r}^2 + 2\gamma^4r^2 + 2\gamma^2(1-\gamma^2)r^2\hat{r}^2 + \gamma^2r^4 + (1-\gamma^2)r^4\hat{r}^2$$

and

(2.65)

$$N_2 = -\left[\gamma^6 + \gamma^4(1-\gamma^2)r^2 + 2\gamma^4\hat{r}^2 + 2\gamma^2(1-\gamma^2)r^2\hat{r}^2 + \gamma^2\hat{r}^4 + (1-\gamma^2)\hat{r}^4r^2\right].$$

Cancelling like terms in equations (2.64) and (2.65) leaves us with

$$\begin{aligned} N_1 + N_2 = {} & \gamma^4(1-\gamma^2)\hat{r}^2 + 2\gamma^4r^2 + \gamma^2r^4 + (1-\gamma^2)r^4\hat{r}^2 \\ & - \left[\gamma^4(1-\gamma^2)r^2 + 2\gamma^4\hat{r}^2 + \gamma^2\hat{r}^4 + (1-\gamma^2)\hat{r}^4r^2\right]. \end{aligned} \tag{2.66}$$

Observe that

$$\begin{aligned} & \gamma^4(1-\gamma^2)\hat{r}^2 + 2\gamma^4 r^2 - \gamma^4(1-\gamma^2)r^2 - 2\gamma^4\hat{r}^2 \\ &= (-\gamma^4+\gamma^6+2\gamma^4)r^2 - (-\gamma^4+\gamma^6+2\gamma^4)\hat{r}^2 \quad (2.67) \\ &= \gamma^4(1+\gamma^2)r^2 - \gamma^4(1+\gamma^2)\hat{r}^2 \quad (2.68) \\ &= (r^2-\hat{r}^2)\gamma^4(1+\gamma^2). \quad (2.69) \end{aligned}$$

Furthermore, note that

$$\begin{aligned} \gamma^2 r^4 + \gamma^2 \hat{r}^4 &= \gamma^2(r^4+\hat{r}^4) \quad (2.70) \\ &= \gamma^2(r^2-\hat{r}^2)(r^2+\hat{r}^2). \quad (2.71) \end{aligned}$$

Substituting the expressions given by equations (2.69) and (2.71) into equation (2.66), we obtain

$$\begin{aligned} N_1 + N_2 &= (1-\gamma^2)r^4\hat{r}^2 - (1-\gamma^2)\hat{r}^4 r^2 \quad (2.72) \\ &\quad +(r^2-\hat{r}^2)\gamma^4(1+\gamma^2) + (r^2-\hat{r}^2)(r^2+\hat{r}^2)\gamma^2. \end{aligned}$$

After factoring $(r^2-\hat{r}^2)$ from equation (2.72), we are left with

$$N_1 + N_2 = M_2\bar{M}_2 = (r^2-\hat{r}^2)\left[(1-\gamma^2)r^2\hat{r}^2 + \gamma^4(1+\gamma^2) + \gamma^2(r^2+\hat{r}^2)\right]. \quad (2.73)$$

Returning to ψ in equation (2.52), we can factor $(r^2-\hat{r}^2)$ to achieve

$$\psi = (r^2-\hat{r}^2)\left[\gamma^2 + \frac{(1-\gamma^2)r^2\hat{r}^2 + \gamma^4(1+\gamma^2) + \gamma^2(r^2+\hat{r}^2)}{\bar{M}_2}\right]. \quad (2.74)$$

Substituting the expressions given by equations (2.74) and (2.56) into equation (2.51), we see that all of H can be written as

(2.75)

$$H = \Delta = \frac{(1-\gamma^2)(r^2-\hat{r}^2)^2}{(\sqrt{\omega}+\sqrt{\hat{\omega}})(\gamma^2+\sqrt{\omega})(\gamma^2+\sqrt{\hat{\omega}})}\left[\gamma^2 + \frac{(1-\gamma^2)r^2\hat{r}^2 + \gamma^4(1+\gamma^2) + \gamma^2(r^2+\hat{r}^2)}{(\gamma^2+r^2)\sqrt{\hat{\omega}} + (\gamma^2+\hat{r}^2)\sqrt{\omega}}\right].$$

We have now arrived at the stated formula for χ when $z, \hat{z} \in \mathbb{C}$.

Remark 4. Observe that the chordal metric $\chi \geq 0$ gives us the inequality

$$F(D^2-\Delta) \geq 0 \Rightarrow D^2 \geq \Delta.$$

Remark 5. Note that $(1-\gamma^2)$ appears as a factor at the front of Δ. For $\gamma^2 = 1$, $\Delta = 0$ and

$$F = \frac{4}{(1+r^2)(1+\hat{r}^2)},$$

leaving us with the Reimann Sphere chordal metric.

Next, when $z \in \text{ID}$ and $\hat{z} \in \mathbb{C}$, we shall derive the formula stated in Theorem 2.1 by observing the asymptotic behavior of χ as $r \to \infty$.

First, observe that

$$\begin{aligned} \omega &= \gamma^2 + (1-\gamma^2)r^2 \quad (2.76) \\ &= (1-\gamma^2)r^2\left[1 + \frac{\gamma^2}{(1-\gamma^2)r^2}\right] \quad (2.77) \\ &\sim (1-\gamma^2)r^2 \text{ as } r \to \infty. \quad (2.78) \end{aligned}$$

Similarly,

$$\hat{\omega} \sim (1-\gamma^2)\hat{r}^2 \text{ as } \hat{r} \to \infty. \quad (2.79)$$

Furthermore, observe that

$$\frac{x}{r} \sim \cos\theta, \qquad \frac{y}{r} \sim \sin\theta, \tag{2.80}$$

and

$$\frac{\hat{x}}{r} \sim \cos\hat{\theta}, \qquad \frac{\hat{y}}{r} \sim \sin\hat{\theta} \tag{2.81}$$

as $r, \hat{r} \to \infty$.

Substituting the asymptotic expression for ω given in equation (2.78) into equation (2.13) and multiplying the first term inside the bracket by $\frac{r}{r}$, we see that

$$\begin{aligned} &\|Z - \hat{Z}\|^2 \\ &\sim \; 2 \\ &\quad -2\left[\left(\frac{\gamma^2 + r\sqrt{1-\gamma^2}}{\gamma^2 + r^2}\right)\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right) r\left(\frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{\gamma^2}{r}\right)\right. \\ &\quad \left. + \gamma^2 - \gamma^2\left(\frac{\gamma^2 + r\sqrt{1-\gamma^2}}{\gamma^2 + r^2}\right) - \gamma^2\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)\right] \end{aligned} \tag{2.82}$$

as $r \to \infty$.

Substituting the asymptotic expressions for $\frac{x}{r}$ and $\frac{y}{r}$ given in equation (2.80) into equation (2.82) and factoring powers of r yields

$$\begin{aligned} &\|Z - \hat{Z}\|^2 \\ &\sim \; 2 \\ &\quad -2\left[\frac{r\sqrt{1-\gamma^2}\left(1 + \frac{\gamma^2}{r\sqrt{1-\gamma^2}}\right)}{r^2\left(1 + \frac{\gamma^2}{r^2}\right)}\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right) r\left(\hat{x}\cos\theta + \hat{y}\sin\theta + \frac{\gamma^2}{r}\right)\right. \\ &\quad \left. + \gamma^2 - \gamma^2\frac{r\sqrt{1-\gamma^2}\left(1 + \frac{\gamma^2}{r\sqrt{1-\gamma^2}}\right)}{r^2\left(1 + \frac{\gamma^2}{r^2}\right)} - \gamma^2\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)\right] \end{aligned} \tag{2.83}$$

$$\sim \; 2 - 2\left[\sqrt{1-\gamma^2}\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)(\hat{x}\cos\theta + \hat{y}\sin\theta) + \gamma^2 - \gamma^2\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)\right] \tag{2.84}$$

$$\sim \; 2\gamma^2\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right) + 2(1-\gamma^2) - 2\sqrt{1-\gamma^2}\left(\frac{\gamma^2 + \sqrt{\hat{\omega}}}{\gamma^2 + \hat{r}^2}\right)(\hat{x}\cos\theta + \hat{y}\sin\theta), \tag{2.85}$$

as $r \to \infty$.

We have now arrived at the stated formula for χ when $z \in \mathrm{ID}, \hat{z} \in \mathbb{C}$.

Remark 6. Note that for $\gamma^2 = 1$,

$$\chi^2(z,\ \hat{z}) = 2\frac{1+1}{1+\hat{r}^2} + 0 = \frac{4}{1+\hat{r}^2},$$

leaving us with the Reimann Sphere chordal metric.

Finally, we shall derive the formula for χ when both $z, \hat{z} \in \mathrm{ID}$, as stated in Theorem 2.1. We shall do this by observing the asymptotic behavior of χ as both $r, \hat{r} \to \infty$.

We begin by substituting the asymptotic expression for $\hat{\omega}$ given in equation (2.79) into equation (2.85) and multiplying the third term by $\frac{\hat{r}}{\hat{r}}$. This leads us to the asymptotic expression

$$\|Z-\hat{Z}\|^2 \tag{2.86}$$
$$\sim\ 2\gamma^2\left(\frac{\gamma^2+\hat{r}\sqrt{1-\gamma^2}}{\gamma^2+\hat{r}^2}\right)+2(1-\gamma^2)-2\sqrt{1-\gamma^2}\left(\frac{\gamma^2+\hat{r}\sqrt{1-\gamma^2}}{\gamma^2+\hat{r}^2}\right)\hat{r}\left(\frac{\hat{x}}{\hat{r}}\cos\theta+\frac{\hat{y}}{\hat{r}}\sin\theta\right)$$

as $r, \hat{r} \to \infty$.

Substituting the asymptotic expressions for $\frac{\hat{x}}{\hat{r}}$ and $\frac{\hat{y}}{\hat{r}}$ given in equation (2.81) into equation (2.86) and factoring powers of $\hat{r}$ yields

$$\|Z-\hat{Z}\|^2$$
$$\sim\ 2\gamma^2\frac{\hat{r}\sqrt{1-\gamma^2}\left(1+\frac{\gamma^2}{\hat{r}\sqrt{1-\gamma^2}}\right)}{\hat{r}^2\left(1+\frac{\gamma^2}{\hat{r}^2}\right)}+2(1-\gamma^2) \tag{2.87}$$
$$-2\sqrt{1-\gamma^2}\frac{\hat{r}\sqrt{1-\gamma^2}\left(1+\frac{\gamma^2}{\hat{r}\sqrt{1-\gamma^2}}\right)}{\hat{r}^2\left(1+\frac{\gamma^2}{\hat{r}^2}\right)}\hat{r}\left(\cos\theta\cos\hat{\theta}+\sin\theta\sin\hat{\theta}\right)$$
$$\sim\ 2(1-\gamma^2)-2(1-\gamma^2)\left(\cos\theta\cos\hat{\theta}+\sin\theta\sin\hat{\theta}\right) \tag{2.88}$$
$$\sim\ 2(1-\gamma^2)\left[1-\left(\cos\theta\cos\hat{\theta}+\sin\theta\sin\hat{\theta}\right)\right] \tag{2.89}$$
$$\sim\ 2(1-\gamma^2)\left[1-\cos(\theta-\hat{\theta})\right] \tag{2.90}$$
$$\sim\ 4(1-\gamma^2)\sin^2\left(\frac{\theta-\hat{\theta}}{2}\right) \tag{2.91}$$

as $r, \hat{r} \to \infty$.

We have now arrived at the stated formula for χ when $z, \hat{z} \in \mathrm{ID}$.

Remark 7. As we expect, for $\gamma^2 = 1$,

$$\chi\left(\infty(\cos\theta,\sin\theta),\ \infty(\cos\hat{\theta},\sin\hat{\theta})\right)=0.$$

On the Reimann sphere, all arguments of infinity map to the north pole.

We are left with the task of proving the completeness of $\mathbb{C} \cup \mathrm{ID}$ in the metric $\chi(z, \hat{z})$. To this end we consider a sequence $Z_n = (x_{1n}, x_{2n}, x_{3n})$ that converges to some $Z = (x_1, x_2, x_3)$. We must show that whenever Z_n converges to Z, there exists a $z \in \mathbb{C} \cup \mathrm{ID}$ such that $G^{-1}(Z_n) = z_n$ converges to $G^{-1}(Z) = z$.

We consider two cases, $x_3 < \gamma$ and $x_3 = \gamma$. Recall that $x_{3n} = \gamma(1 - t_n)$.

Case 1. For $x_3 < \gamma$,

$$x_{3n}=\gamma(1-t_n)\Leftrightarrow t=1-\frac{x_3}{\gamma}. \tag{2.92}$$

Because $x_3 < \gamma$, we know that $t \neq 0$ and therefore

$$z_n = (x_n, y_n) = \left(\frac{x_{1n}}{t_n}, \frac{x_{2n}}{t_n}\right) \to z = \left(\frac{x_1}{t}, \frac{x_2}{t}\right) \text{ as } n \to \infty. \tag{2.93}$$

Case 2. For $x_3 = \gamma$, we know that $t_n \to 0$ as $n \to \infty$. More specifically, we know that

$$t_n \sim \frac{\sqrt{1-\gamma^2}}{r_n}[1 + U_n] \text{ as } r_n \to \infty, \tag{2.94}$$

where $U_n \to 0$ as $r_n \to \infty$.

Since Z is a point on the unit sphere,

$$x_{1n}^2 + x_{2n}^2 = 1 - x_{3n}^2 \to 1 - \gamma^2 \text{ as } n \to \infty, \tag{2.95}$$

and

$$\left(\frac{x_1}{\sqrt{1-\gamma^2}}\right)^2 + \left(\frac{x_2}{\sqrt{1-\gamma^2}}\right)^2 = 1. \tag{2.96}$$

Then we see that

$$\frac{x_1}{\sqrt{1-\gamma^2}} = \cos\theta \text{ and } \frac{x_2}{\sqrt{1-\gamma^2}} = \sin\theta \text{ for some } 0 \leq \theta < 2\pi. \tag{2.97}$$

Furthermore, as $n \to \infty$, we see from equation (2.94) that

$$\frac{x_n}{r_n} = \frac{x_{1n}}{r_n t_n} \sim \frac{x_{1n}}{\sqrt{1-\gamma^2}[1+U_n]} \to \cos\theta \quad \text{and} \quad \frac{y_n}{r_n} \to \sin\theta. \tag{2.98}$$

Recall the definition of convergence to a point in ID given in Definition 1.5. With this definition in mind, we see that

$$z_n = r_n \left(\frac{x_n}{r_n}, \frac{y_n}{r_n}\right) \to \infty(\cos\theta,\ \sin\theta) \text{ as } n \to \infty. \tag{2.99}$$

Therefore, whenever Z_n converges to Z, z_n converges to some $z \in \mathbb{C} \cup \text{ID}$.

□

Remark 8. The compacitification and induced metric presented here can be extended for $\mathbb{R}^n$ projected onto a "bowl" in $\mathbb{R}^{n+1}$. Then the induced metric is defined analagously to Equation (2.1), where z and $\hat{z}$ are two points in $\mathbb{R}^n$. Consequently, the expression of the metric in Equation (2.2) is precisely the same for $z, \hat{z} \in \mathbb{R}^n$, where r^2 and $\hat{r}^2$ retain the same meaning as the Euclidean distance of a point to the origin in $\mathbb{R}^n$ and D^2 is the square of the Euclidean distance between z and $\hat{z}$,

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